

Writing $X_h = \sum_{j=0}^m a_{jh}x_j$ ($h = 0, 1, \dots, n$), it is familiar that $M(\alpha, \beta)$ is also the maximum of the ratio $(\sum_0^n |X_h|^{1/\beta'})^{\beta'}/[\int_0^{2\pi} |f(e^{i\theta})|^{1/\alpha}]^\alpha$, where $\beta + \beta' = 1$. From sums of powers of linear forms we pass, by the classical process, to integrals, and the theorem of Littlewood and Paley is the consequence of the inequality (2) applied to the two extreme cases $\alpha = \beta' = 1/2$ ($p = 2$) and $\alpha = \beta' = 1$ ($p = 1$).

Remark: In our main theorem, the condition on the y 's can be replaced by one similar to that imposed on the x 's. But this has no interesting application. By passages to limits we can also obtain results about linear operations from H^p to L^q , where $p, q \geq 1$.

¹ Littlewood and Paley, *Proc. London Math. Soc.*, **43**, 105 (1937).

² For a simple proof in the case of $p = 2$, see Hardy and Littlewood, *Proc. Cambridge Phil. Soc.*, **40**, 103-107 (1938).

³ See Zygmund, A., *Fundamenta Mathematicae*, **30**, 190 (1938).

⁴ A function $f(z) = \sum c_n z^n$, regular for $|z| < 1$, is said to belong to the class H^p if $\int_0^{2\pi} |f(re^{i\theta})|^p d\theta$ remains bounded as $r \rightarrow 1$. For the classical results of the theory see, e.g., Zygmund, *Trigonometrical Series*, Chapter VII.

⁵ Thorin, G. O., *Convexity Theorems*, Uppsala, 1948, pp. 1-57, esp. 31-35.

⁶ Thorin, G. O., "An Extension of a Convexity Theorem Due to M. Riesz," *Kungl. Fysiografiska Sällskapet i Lund Förhandlingar*, **8** (1939), nr. 14. Tamarkin, J. D., and Zygmund, A., "Proof of a Theorem of Thorin," *Bull. the Am. Math. Soc.*, **50**, 279-282 (1944). Salem, R., Sur une extension du théorème de convexité de M. Marcel Riesz, *Colloquium Mathematicum*, Wroclaw, 1947, Vol. I, pp. 6-8.

⁷ In the case $\beta = 0$, the condition is to be interpreted, as usual, as $\max |y_h| \leq 1$.

⁸ Thorin's idea (see footnote 5) of taking for f the k th power of an analytic function and applying his convexity theorem, is basic for the whole argument. If we wanted to restrict ourselves to the proof of the Littlewood-Paley result, the argument could be simplified still further, since here we interpolate between $\alpha = 1$ and $\alpha = 1/2$, and we could imitate more closely the proof of Thorin (see footnote 5, pp. 31-35).

⁹ See e.g. Marcinkiewicz and Zygmund, *Fundamenta Mathematicae*, **28**, 131-166 (1937).

NOTES ON INTEGRATION, II

By M. H. STONE

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF CHICAGO

Communicated July 6, 1946

The results of our first note¹ enable us to treat the following classes of functions, all contained in \mathfrak{G} :

$$\mathfrak{F}_p = \{f; |f|^{p-1} \in \mathfrak{F}\} = \{f; N(|f|^p) < +\infty\}, \quad \mathfrak{F}_p = \{f; |f|^{p-1} \in \mathfrak{F}\}, \\ \mathfrak{M} = \{f; \text{mid}(f, g, h) \in \mathfrak{F} \text{ for all } g \text{ and } h \text{ in } \mathfrak{F}\},$$

where $p \geq 1$ and $\text{mid}(\lambda, \mu, \nu)$ designates the intermediate one of the three numbers λ, μ, ν in accordance with the precise relations

$$\begin{aligned}\text{mid}(\lambda, \mu, \nu) &= \max(\min(\lambda, \mu), \min(\mu, \nu), \min(\nu, \lambda)) \\ &= \min(\max(\lambda, \mu), \max(\mu, \nu), \max(\nu, \lambda)).\end{aligned}$$

Obviously, $\mathfrak{F}_1 = \mathfrak{F}$ and $\mathfrak{L}_1 = \mathfrak{L}$. The importance of the class \mathfrak{L}_p is well established, and the consideration of \mathfrak{F}_p along with \mathfrak{L}_p is natural. Since in the classical instances of our theory \mathfrak{M} can be identified as the totality of measurable functions, we shall call any function in \mathfrak{M} a *measurable function*.

In order to discuss \mathfrak{F}_p we need to establish for the quantity $N_p(f) = N(|f|^p)^{1/p}$ the inequalities

- (1) (Hölder) if $p > 1$ and $p + q = pq$, then $N(fg) \leq N_p(f)N_q(g)$;
- (2) (Minkowski) $N_p(f + g) \leq N_p(f) + N_p(g)$.

The proof of (1) begins with the observation that for $\alpha > 0, \beta > 0$, and $\gamma = 1/[(q/p)^{1/q} + (p/q)^{1/p}]$ the function $\gamma(\alpha\xi^p + \beta\xi^{-q})$, $0 < \xi < +\infty$, has $\alpha^{1/p}\beta^{1/q}$ as its absolute minimum and assumes this value only when $\xi = \xi_0 = (\beta q/\alpha p)^{1/pq}$. As a result we see that $|fg| \leq \gamma(|f|^p\xi^p + |g|^q\xi^{-q})$ and hence that $N(fg) \leq \gamma(N(|f|^p)\xi^p + N(|g|^q)\xi^{-q})$ for all $\xi > 0$. On putting $N(|f|^p) = \alpha$, $N(|g|^q) = \beta$, and $\xi = \xi_0$ in the latter inequality we obtain (1). Since for $p = 1$ the inequality (2) has already been established (as a special case of I (7)), we suppose that $p > 1$. The proof for this case then begins with the observation that $|\xi + \eta|^p \leq 2^{p-1}(|\xi|^p + |\eta|^p)$. We therefore have $N(|f + g|^p) \leq 2^{p-1}(N(|f|^p) + N(|g|^p))$, so that $f + g$ is in \mathfrak{F}_p whenever f and g are. Consequently we can use (1) and the relation $|f + g|^p \leq |f||f + g|^{p-1} + |g||f + g|^{p-1}$ to obtain

$$\begin{aligned}N(|f + g|^p) &\leq N(|f||f + g|^{p-1}) + N(|g||f + g|^{p-1}) \\ &\leq (N_p(f) + N_p(g))N(|f + g|^p)^{1/q}\end{aligned}$$

an inequality from which (2) follows at once. The Minkowski inequality shows immediately that identification of functions f and g for which $N_p(f - g) = 0$ will permit us to treat \mathfrak{F}_p as a real normed vector-lattice with N_p as its norm-function. It is easily verified that this identification is the same for all values of p , two functions being identified if and only if they are equal almost everywhere. We can now generalize I (10) to read

- (3) the normed vector space \mathfrak{F}_p is complete (and hence a Banach space).

The proof will be sketched for $p > 1$. By definition the mapping Φ which carries f into $g = |f||f|^{p-1} = |f|^p \text{sgn } f$ maps \mathfrak{F}_p onto \mathfrak{F} . It has as its inverse the mapping Ψ which carries g into $f = |g|^{1/p} \text{sgn } g$. The local

behavior of these mappings can be determined by an appeal to the inequalities²

$$2^{1-p}|\xi - \eta|^p \leq |\xi|\xi|^{p-1} - \eta|\eta|^{p-1}| \leq p|\xi - \eta|(|\xi|^{p-1} + |\eta|^{p-1}),$$

which hold even for complex ξ and η . The first yields $|\Psi(g_1) - \Psi(g_2)|^p \leq 2^{p-1}|g_1 - g_2|^p$ and hence $N_p(\Psi(g_1) - \Psi(g_2)) \leq 2^{1/q}N(g_1 - g_2)^{1/p}$ for all g_1 and g_2 in \mathfrak{F} . Accordingly Ψ is continuous, belonging in fact to the Lipschitz class $\text{Lip } 1/p$. Similarly, the second inequality yields $|\Phi(f_1) - \Phi(f_2)| \leq p(|f_1 - f_2||f_1|^{p-1} + |f_1 - f_2||f_2|^{p-1})$ and hence, with the help of the Hölder inequality,

$$N(\Phi(f_1) - \Phi(f_2)) \leq pN_p(f_1 - f_2)(N_p(f_1)^{p/q} + N_p(f_2)^{p/q}).$$

Accordingly Φ is continuous, belonging on any bounded part of \mathfrak{F}_p to the associated Lipschitz class $\text{Lip } 1$. If now $\{f_n\}$ is a Cauchy sequence in \mathfrak{F}_p , it is bounded and must therefore be carried by Φ into a Cauchy sequence $\{g_n\}$, $g_n = \Phi(f_n)$, in \mathfrak{F} . The completeness of \mathfrak{F} shows that the latter sequence has a limit g in \mathfrak{F} . Thus the function $f = \Psi(g)$ is the limit of $\{f_n\}$ in \mathfrak{F}_p , by virtue of the continuity of Ψ . This completes the proof. We note further that we have obtained at the same time the following variant of a result of Mazur:²

(4) *The spaces \mathfrak{F}_p , $p \geq 1$, are mutually homeomorphic and all have the same linear dimension.*

The first part of (4) follows from the fact that Φ maps \mathfrak{F}_p homeomorphically onto \mathfrak{F} , as shown above. The second results from a topological interpretation of the linear dimension. It is well known that a Banach space has finite linear dimension if and only if it is locally compact, in which case it is homeomorphic to an n -dimensional Euclidean space and its linear dimension is n . On the other hand, if the linear dimension of a Banach space is infinite it is equal to the density-character of the space—that is, to the least among the cardinal numbers of everywhere dense parts of the space.

Turning now to the consideration of \mathfrak{L}_p , we proceed to specialize and sharpen the results of the previous paragraph. First we replace (1) by the more detailed statement

- (1') (Hölder) *if $f \in \mathfrak{L}_p$ and $g \in \mathfrak{L}_q$, then $fg \in \mathfrak{L}$ and $|L(fg)| \leq L(|f|^p)^{1/p} \cdot L(|g|^q)^{1/q}$, the equality holding if and only if $f|f|^{p-1}$ and $g|g|^{q-1}$ are linearly dependent in \mathfrak{L} (when $p = q = 2$, if and only if f and g are linearly dependent in \mathfrak{L}_2).*

To show that $fg \in \mathfrak{L}$ under the present hypotheses, we appeal to I (13), writing $fg = \varphi(f', g')$ where $f' = f|f|^{p-1} \in \mathfrak{L}$, $g' = g|g|^{q-1} \in \mathfrak{L}$, and $\varphi(\lambda, \mu) = |\lambda|^{1/p}|\mu|^{1/q} \operatorname{sgn} \lambda\mu$. Thus we have

$$|L(fg)| \leq L(|fg|) = N(fg) \leq N_p(f)N_q(g) = L(|f|^p)^{1/p}L(|g|^q)^{1/q}$$

by virtue of (1). In determining the conditions under which the extreme terms here are equal, we may discard the trivial case where the last term vanishes, one of the two functions f and g being a null function. The indicated equality is equivalent to the continued equation $\pm L(fg) = L(|fg|) = L(|f|^p)\xi_0^p + L(|g|^q)\xi_0^{-q} = L(|f|^p\xi_0^p + |g|^q\xi_0^{-q})$, where ξ_0 has the value indicated in the proof of (1) and a fixed determination of the ambiguous sign is adopted for the remainder of the discussion. In view of the relations $\pm fg \leq |fg| \leq |f|^p\xi_0^p + |g|^q\xi_0^{-q}$, the continued equation for the integrals holds if and only if $\pm fg = |fg| = |f|^p\xi_0^p + |g|^q\xi_0^{-q}$ almost everywhere. The second equation here holds if and only if $|g|^q = (p/q)\xi_0^q|f|^p$ almost everywhere. Consequently both equations hold if and only if $g|g|^{q-1} = \pm(p/q)\xi_0^q|f|^{p-1}$ almost everywhere. The proof of (1') is thereby completed. In a quite similar fashion we find that (2) can be replaced by

- (2') (Minkowski) *if f and g are in \mathfrak{L}_p , then $f + g$ is in \mathfrak{L}_p and $L(|f + g|^p)^{1/p} \leq L(|f|^p)^{1/p} + L(|g|^p)^{1/p}$ the equality holding if and only if f and g are linearly dependent in \mathfrak{L}_p and of the same sign almost everywhere.*

To show that $f + g \in \mathfrak{L}_p$ under the present hypotheses, we appeal to I (13), writing $(f + g)|f + g|^{p-1} = \varphi(f', g')$ where $f' = f|f|^{p-1} \in \mathfrak{L}$, $g' = g|g|^{p-1} \in \mathfrak{L}$, and $\varphi(\lambda, \mu) = (|\lambda|^{1/p} \operatorname{sgn} \lambda + |\mu|^{1/p} \operatorname{sgn} \mu) \{ |\lambda|^{1/p} \operatorname{sgn} \lambda + |\mu|^{1/p} \operatorname{sgn} \mu \}^{p-1}$. The result given in (2) now assumes the present form. Reviewing the proof of (2) under the conditions postulated here we see that the equality holds if and only if $|f + g| = |f| + |g|$ almost everywhere while $|f|^p$, $|g|^p$, and $|f + g|^p$ are linearly dependent in \mathfrak{L} in harmony with (1'); but these conditions are equivalent to those stated above. Since \mathfrak{L}_p clearly contains αf and $|f|$ together with f , it is a linear sublattice of \mathfrak{F}_p . Moreover the fact that the homeomorphic mapping Φ carries \mathfrak{L}_p onto \mathfrak{L} , where \mathfrak{L} is closed in \mathfrak{F} , shows that \mathfrak{L}_p is closed in \mathfrak{F}_p . Accordingly we have

- (3') \mathfrak{L}_p is a closed linear subspace of \mathfrak{F}_p (and hence is a Banach space).

We can give a rather wide extension of (4) which will cover not only the spaces \mathfrak{L}_p , $p \geq 1$, arising from a fixed elementary integral but also the spaces \mathfrak{L}_p arising from different elementary integrals. This is possible because the space \mathfrak{L}_2 is a generalized Euclidean or real Hilbert space, its norm being obtained in the appropriate manner from the integral $L(fg)$, which depends linearly and symmetrically upon f and g . Two such spaces are homeomorphic if and only if they have the same linear dimension, as is well known. Hence we have

- (4') the spaces \mathfrak{L}_p , $p \geq 1$, arising from a fixed elementary integral are mutually homeomorphic and all have the same linear dimension; two such spaces arising from different elementary integrals are homeomorphic if and only if they have the same linear dimension.

This result completes our discussion of the spaces \mathfrak{L}_p .

In the study³ of the class \mathfrak{M} , it is convenient to observe that $\text{mid}(\lambda, \mu, \nu)$ is a positively homogeneous, continuous function satisfying the inequalities $\min(\mu, \nu) \leq \text{mid}(\lambda, \mu, \nu) \leq \max(\mu, \nu)$, since we thereby justify applications of I (13) and of the dominated-convergence theorem at various points below. It is, of course, true that $\text{mid}(\lambda, \mu, \nu)$ has an explicit expression in terms of λ, μ, ν by means of addition, multiplication by constants and formation of absolute values. Thus we can show at once that

- (5) if $f = \lim_{n \rightarrow \infty} f_n$ and $f_n \in \mathfrak{M}$, then $f \in \mathfrak{M}$.

In fact we have $\text{mid}(f_n, g, h) \in \mathfrak{L}$ whenever g and h are in \mathfrak{L} , and hence conclude by the dominated-convergence theorem that $\text{mid}(f, g, h) = \lim_{n \rightarrow \infty} \text{mid}(f_n, g, h)$ is also in \mathfrak{L} . We also have

- (6) if f_1, \dots, f_m are in \mathfrak{M} and if $\varphi(\lambda_1, \dots, \lambda_m)$ is any positively homogeneous function of Baire (bounded or not) defined for $-\infty \leq \lambda_k \leq +\infty$ then $\varphi(f_1, \dots, f_m) \in \mathfrak{M}$; in particular, the functions $\alpha f, |f|, f + g$ are measurable whenever f and g are.

Once the indicated particular cases have been handled, the general case is obtained by passages to the limit similar to those used in deriving I (13) from I (11). Taking the case of $|f|$ as typical, we wish to show that $\text{mid}(|f|, g, h) \in \mathfrak{L}$ whenever g and h are in \mathfrak{L} . By hypothesis the function $f_n = \text{mid}(f, n(|g| + |h|), -n(|g| + |h|))$ is in \mathfrak{L} ; and so also is the function $g_n = \text{mid}(|f_n|, g, h)$. It is easily seen that $\lim_{n \rightarrow \infty} f_n(x)$ has the value 0 or the value $f(x)$ according as $|g(x)| + |h(x)|$ vanishes or not. Hence $\lim_{n \rightarrow \infty} g_n = \text{mid}(|f|, g, h)$, and the latter function is in \mathfrak{L} by the dominated-convergence theorem. In practice it is convenient to have available the following criterion:

- (7) when $f \geq 0$, a necessary and sufficient condition for f to be measurable is that $\min(f, g) \in \mathfrak{L}$ whenever $g \in \mathfrak{L}$ and $g \geq 0$.

The necessity of the stated condition results immediately from the identity $\min(f, g) = \text{mid}(f, g, 0)$. The sufficiency is established by noting that for arbitrary g and h in \mathfrak{L} we have $\text{mid}(f, g, h) = \max(\min(f, \max(g, h)), \min(g, h)) \in \mathfrak{L}$ since $\min(f, \max(g, h)) = \min(f, \max(g, h))$.

0)) + $\min(0, \max(g, h)) \in \mathfrak{L}$ by hypothesis. From (7) and I (14) we have

- (8) *the constant function 1 is measurable if and only if $\min(1, g)$ is integrable whenever g is; and it is measurable whenever I (3) is verified.*

Hence we immediately obtain the following generalization of (6) above

- (9) *if $1, f_1, \dots, f_m$ are measurable and if $\varphi(\lambda_1, \dots, \lambda_m)$ is any function of Baire (bounded or not) defined for $-\infty \leq \lambda_k \leq +\infty$, then $\varphi(f_1, \dots, f_m) \in \mathfrak{M}$.*

We conclude our general remarks on measurable functions by noting some criteria for integrability. First we have

- (10) *in order that a measurable function f be integrable it is necessary and sufficient that $N(f) < +\infty$; and it is likewise necessary and sufficient there exist integrable functions g and h , such that $g \leq f \leq h$.*

If f is integrable, then $N(f) < +\infty$ and the second condition is satisfied with $g = h = f$. If $N(f) < +\infty$ we can find a function $k \in \mathfrak{L}$ such that

$|f| \leq k$, since there exist elementary functions f_n such that $|f| \leq \sum_{n=1}^{\infty} |f_n|$, $\sum_{n=1}^{\infty} E(|f_n|) < +\infty$, and the function $k = \sum_{n=1}^{\infty} |f_n|$ is in \mathfrak{L} by virtue of I (12).

Thus $N(f) < +\infty$ implies that $g \leq f \leq h$ where $g = -k$ and $h = k$ are in \mathfrak{L} . When there exist such g and h , we have $f = \text{mid}(f, g, h) \in \mathfrak{L}$ by the definition of \mathfrak{M} . Since any integrable function is clearly measurable, by direct application of the definition, we see that (10) yields the relations

$$(11) \quad \mathfrak{L} = \mathfrak{M} \cap \mathfrak{F} = \{f; f \in \mathfrak{M}, N(f) < +\infty\}.$$

It follows immediately that

$$(12) \quad \mathfrak{L}_p = \{f; |f|^{p-1} \in \mathfrak{M}, N_p(f) < +\infty\}.$$

When 1 is measurable we can give the last result the sharper form,

$$(13) \quad \text{if } 1 \in \mathfrak{M}, \text{ then } \mathfrak{L}_p = \mathfrak{M} \cap \mathfrak{F}_p = \{f; f \in \mathfrak{M}, N_p(f) < +\infty\},$$

since (9) shows that the functions $g = |f|^{p-1}$ and $f = |g|^{1/p} \text{sgn } g$ are measurable together.

Finally we shall consider the connections between our theory of the general integral and the theory of measure. By specializing I (5)–(7) we see that an outer measure μ^* is defined for the subsets of X by putting $\mu^*(Y) = N(f_Y)$ where f_Y is the characteristic function of $Y \subset X$. Similarly, we see from the properties of the general integral that the finite set-function μ defined by putting $\mu(Y) = L(f_Y)$ whenever $f_Y \in \mathfrak{L}$ is a *completely additive*

measure. It is natural to define a set Y to be measurable if and only if $f_Y \in \mathfrak{M}$; but we must then consider how such sets are related to the μ^* -measurable sets Y , characterized by the now classical condition that $\mu^*(Z) = \mu^*(Z \cap Y) + \mu^*(Z \cap Y')$ for all Z . A fundamental tool in this investigation is provided by the following result:

- (14) *if $N(f) < +\infty$, there exists a function $g \in \mathfrak{L}$ such that $|f| \leq g$ and $N(f) = L(g)$ —in particular, if $1 \in \mathfrak{M}$ and f is a characteristic function, then g may be chosen to be a characteristic function.*

Adopting a device used in the proof of (10) above, we select elementary functions f_{mn} such that $|f| \leq \sum_{n=1}^{\infty} |f_{mn}| = g_m$, $N(f) \leq \sum_{n=1}^{\infty} E(|f_{mn}|) \leq N(f) + 1/m$ and note that $g_m \in \mathfrak{L}$ and $N(f) \leq L(g_m) \leq N(f) + 1/m$ in accordance with I (12). It is then clear that $g = \lim_{m \rightarrow \infty} \min(g_1, \dots, g_m)$ has the desired properties. In the indicated special case, we may first suppose that $g \leq 1$ since otherwise we can replace g by $\min(1, g) \in \mathfrak{L}$ in accordance with (8); and we may then suppose that g is a characteristic function, since otherwise we can replace g by the characteristic function $\lim_{n \rightarrow \infty} g^n \in \mathfrak{L}$, in accordance with (9), (10) and the dominated-convergence theorem. From (14) we obtain

- (15) *every measurable set is μ^* -measurable; but the converse is true if and only if $1 \in \mathfrak{M}$.*

We sketch the proof. Assuming Y to be measurable, we have to reduce the inequality $\mu^*(Z) \leq \mu^*(Z \cap Y) + \mu^*(Z \cap Y')$ to an equality. Since the reduction is automatic when $\mu^*(Z) = +\infty$, we suppose that $\mu^*(Z) < +\infty$. Then (14) furnishes a function $g \in \mathfrak{L}$ such that $f_Z \leq g$, $\mu^*(f_Z) = L(g)$. Since $f_{Z \cap Y} = f_Z f_Y \leq g f_Y = \min(g, f_Y) \leq g$ we see that $g f_Y \in \mathfrak{L}$, $\mu^*(Z \cap Y) \leq L(g f_Y)$. The inequality $f_{Z \cap Y'} = f_Z(1 - f_Y) \leq g(1 - f_Y) = g - g f_Y \in \mathfrak{L}$ shows that $\mu^*(Z \cap Y') \leq L(g - g f_Y)$. By addition we obtain $\mu^*(Z \cap Y) + \mu^*(Z \cap Y') \leq L(g f_Y) + L(g - g f_Y) = L(g) = \mu^*(Z)$, thereby completing the discussion. Looking now at the converse, we see that X is trivially μ^* -measurable and hence that the converse cannot hold unless $1 = f_X \in \mathfrak{M}$. Assuming this necessary condition, we now show that $f_Y \in \mathfrak{M}$ by (7), whenever Y is μ^* -measurable. Starting with an arbitrary non-negative integrable function g , we put $f_n = \varphi_n(g) \leq ng$, $g_n = g f_n \leq g$ where φ_n is the characteristic function of the interval $1/n \leq \lambda \leq +\infty$. By (9) and (10) we see that f_n and g_n are integrable. Moreover f_n is the characteristic function of a set Z_n with $\mu(Z_n) = L(f_n)$. Since $\min(f_Y, g) = \lim_{n \rightarrow \infty} \min(f_Y, g_n) = \lim_{n \rightarrow \infty} (f_Y f_n, g) \leq g$ it suffices to show that $f_Y f_n \in \mathfrak{L}$. Since $f_Y f_n$ is the characteristic function of $Y \cap Z_n$, the special

case of (14) furnishes us with a set W_n such that $Y \cap Z_n \subset W_n$, $\mu^*(Y \cap Z_n) = \mu(W_n)$. We may suppose that $W_n \subset Z_n$, since otherwise we can replace it by $W_n \cap Z_n$. We now have $\mu(W_n) = \mu^*(W_n) = \mu^*(W_n \cap Y) + \mu^*(W_n \cap Y') \geq \mu^*(W_n \cap Y) = \mu^*(Y \cap Z_n) = \mu(W_n)$. Hence $\mu^*(W_n \cap Y') = 0$ and $W_n \cap Y'$ is a null set. Thus $f_Y f_n$ differs from the characteristic function of W_n by a null function and is integrable. This completes the proof. From (14) and (15) we at once derive

- (16) *when $1 \in \mathfrak{M}$, $\mu^*(Y) < +\infty$ is the minimum of the measures $\mu(Z)$ where $Z \supset Y$; and $\mu(Y)$ exists if and only if Y is μ^* -measurable and has finite outer measure, in which case $\mu(Y) = \mu^*(Y)$.*

On the basis of the preceding results, we can now establish

- (17) *when $1 \in \mathfrak{M}$, a finite function f is measurable if and only if the sets $\{x; \alpha \leq f(x) < \beta\}$ are measurable for all (rational) α and β , $\alpha > \beta$; and, when f is integrable, its general integral $L(f)$ is the limit of the Lebesgue sums*

$$\sigma(f; \epsilon) = \sum_{k=-\infty}^{k=+\infty} \sigma_k \mu\{x; \alpha_k \leq f(x) < \alpha_{k+1}\},$$

where $\lim_{k \rightarrow -\infty} \alpha_k = -\infty$, $\lim_{k \rightarrow +\infty} \alpha_k = +\infty$, $\alpha_k \leq \sigma_k \leq \alpha_{k+1}$, $\alpha_{k+1} - \alpha_k \leq \epsilon$, and $\sigma_k = 0$ for $\min(|\alpha_k|, |\alpha_{k+1}|) < \epsilon$, the summation omitting those terms in which $\sigma_k = 0$.

The measurability of the set $\{x; \alpha \leq f(x) < \beta\}$ is equivalent to that of its characteristic function, expressible as $\varphi_{\alpha\beta}(f)$ where $\varphi_{\alpha\beta}$ is the characteristic function of the interval $\alpha \leq \lambda < \beta$. By (9) this function is measurable when f is. On the other hand, if $\varphi_{\alpha\beta}(f)$ is measurable for all α and β

(or even just for rational α and β), we see that the function $f_\epsilon = \sum_{k=-\infty}^{k=+\infty} \sigma_k \varphi_k(f)$,

where $\varphi_k = \varphi_{\alpha_k \alpha_{k+1}}$, is measurable (for rational α_k , at least) and that its limit $f = \lim_{\epsilon \rightarrow 0} f_\epsilon$ is also, in accordance with (5) and (6). When f is integrable and $\gamma_k = \min(|\alpha_k|, |\alpha_{k+1}|) \geq \epsilon$, we see that $\varphi_k(f) \leq |f|/\gamma_k$,

$|f_\epsilon| \leq 2 \sum_{k=-\infty}^{k=+\infty} \gamma_k \varphi_k(f) \leq 2|f|$ by virtue of the inequality $|\sigma_k| \leq \min(|\alpha_k|, |\alpha_{k+1}|) + \epsilon \leq 2\gamma_k$; and hence that $\varphi_k(f)$ and f_ϵ are both integrable in accordance with (10). The dominated-convergence theorem then yields

$L(f) = \lim_{\epsilon \rightarrow 0} L(f_\epsilon)$, where $L(f_\epsilon) = \sum_{k=-\infty}^{k=+\infty} \sigma_k L(\varphi_k(f)) = \sigma(f; \epsilon)$, as we wished to show.

If we apply these considerations to a particular instance of our general theory we obtain immediately an important theorem:⁴

- (18) *if E is any positive linear functional on the real vector-lattice \mathfrak{E} of*

all continuous real functions with compact nucleus on a locally compact space X , then E can be expressed as an integral in the sense of Lebesgue with respect to a measure on X .

As we have already pointed out, \mathfrak{E} and E satisfy our basic postulates and therefore lead to the introduction of an associated general integral and an associated outer measure. Since I (3) is valid here, we see that $1 \in \mathfrak{M}$ and hence that (15), (16) and (17) are valid also.

It is an open question to determine what modifications (if any) in our definitions and procedures will permit a more thorough analysis of the case where 1 is not measurable.

¹ Stone, M. H., "Notes on Integration, I," these PROCEEDINGS, **34**, 336-342 (1948); cited as I.

² Mazur, S., "Une remarque sur l'homéomorphie des champs fonctionnels," *Studia Math.*, **1**, 83-85 (1929).

³ Mr. H. Rubin, while a member of one of my classes, worked out much useful information on the subject of this paragraph and the next.

⁴ This result is closely related to theorems given by Riesz, F., "Sur certains systèmes singuliers d'équations intégrales," *Ann. Sci. de l'Ec. Norm. Sup.* (3), **28**, 33-62 (1911); Markoff, A., "On Mean Values and Exterior Densities," *Mat. Sbornik*, **4**, 165-191 (1938), especially Theorems 17 and 20; Kakutani, S., "Concrete Representation of Abstract M-Space," *Ann. Math.*, **42**, 994-1024 (1941), especially Theorem 9. The measure found in (18) is always regular in the sense of Carathéodory; but it is defined with certainty only for compact G_δ -sets and not necessarily for all compact sets (except, of course, when X is separable). This measure is therefore not necessarily identical with the one introduced by Markoff. The distinction is that between "Baire measures" and "Borel measures" (in the terminology of P. R. Halmos) and is known to be genuine on the basis of an unpublished example of J. Dieudonné. We shall have more to say about this situation in our fourth note.

SOME PRELIMINARY RESULTS ON THE SPECTRA OF AsH_3 , AsD_3 AND PH_3^*

BY VIRGINIA MARIE McCONAGHIE AND HARALD H. NIELSEN

MENDENHALL LABORATORY OF PHYSICS, OHIO STATE UNIVERSITY, COLUMBUS, OHIO

Communicated by R. S. Mulliken, July 30, 1948

Measurements have been made on several of the bands in the spectra of AsH_3 , AsD_3 and PH_3 . Work is progressing on the analysis of these bands, but since some considerable time will be required to bring this work to completion we wish here to present a report on the work accomplished thus far on the spectra of these molecules.

Figure 1 represents the 4.5μ absorption region in the spectrum of AsH_3 gas. This is the region studied earlier by Lee and Wu¹ under a resolving